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Nehari-Type Oscillation Criteria for Self-Adjoint Linear Differential Equations

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I. INTRODUCTION

Consider the self-adjoint linear differential equation

$$\sum_{k=0}^{n} (-1)^k (p_k(x)\, y^{(k)})^{(k)} = 0, \tag{1.1}$$

where $x \in I = [a, b)$, $b \leqslant \infty$, $p_k \in C^k(I)$, $p_n > 0$ on I. There exists voluminous literature dealing with oscillation properties of Eq. (1.1) and applications of these properties in the spectral analysis of differential operators generated by Eq. (1.1), see [2, 9, 12, 14] and the references given therein.

Recently we proved, using the concept of principal and nonprincipal systems of solutions of (1.1), the following Nehari-type oscillation criterion for a two-term differential equation

$$(-1)^n (p(x)\, y^{(n)})^{(n)} + q(x)\, y = 0. \tag{1.2}$$

THEOREM A [4]. *Let $y_1, ..., y_n$ be the principal (nonprincipal) system of solutions of the equation*

$$(p(x)\, y^{(n)})^{(n)} = 0 \tag{1.3}$$

at b and let (U, V) be the solution of the matrix linear Hamiltonian system corresponding to (1.3) generated by $y_1, ..., y_n$ (for terminology see Section 2). If there exists $c = (c_1, ..., c_n) \in \mathbf{R}^n$ such that for $h = c_1 y_1 + \cdots + c_n y_n$

$$\limsup_{x \to b} \frac{\int_x^b qh^2}{c^T(\int^x U^{-1}BU^{T-1})^{-1} c} < -1 \tag{1.4$_1$}$$

$$\left(\limsup_{x \to b} \frac{\int^x qh^2}{c^T(\int_x^b U^{-1}BU^{T-1})^{-1} c} < -1\right), \tag{1.4$_2$}$$

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where $B = \operatorname{diag}\{0, ..., 0, p^{-1}\}$ *and* ^{T} *denotes the transpose of the matrix indicated, then* (1.2) *is oscillatory at* b.

Another criteria of this kind (they are sometimes called Nehari-type criteria according to Nehari's paper [11]) may be found in [8–10]. These criteria mostly deal with the case $p(x) = x^\alpha$, $\alpha \in \mathbf{R}$, and a typical result is the following criterion of Fiedler [7].

THEOREM B. *Let* $\alpha \notin \{1, 2, ..., 2n-1\}$, $\alpha + \sigma < 2n-1$ $(\alpha + \sigma > 2n-1)$, $\sigma \in \mathbf{R}$, *and*

$$q(x) \leqslant 0 \qquad \textit{for large } x. \tag{1.5}$$

If

$$\liminf_{x \to \infty} x^{2n-1-\alpha-\sigma} \int_x^\infty q(t)\, t^\sigma < -B_{n,\alpha,\sigma} - \frac{\binom{\sigma/2}{n}^2 (n!)^2}{2n-1-\alpha-\sigma} \tag{1.6$_1$}$$

$$\left(\liminf_{x \to \infty} x^{2n-1-\alpha-\sigma} \int_1^x q(t)\, t^\sigma < -\bar{B}_{n,\alpha,\sigma} - \frac{\binom{\sigma/2}{n}^2 (n!)^2}{2n-1-\alpha-\sigma} \right), \tag{1.6$_2$}$$

where $B_{n,\alpha,\sigma}(\bar{B}_{b,\alpha,\sigma})$ *is a real constant depending on* n, α, σ *whose precise value may be found in* [7], *then the equation* $(-1)^n (x^\alpha y^{(n)})^{(n)} + q(x)\, y = 0$ *is oscillatory at* ∞.

Note that in contrast to Theorem B, Theorem A does not require the function q be nonnegative near b. On the other hand, the test function h in (1.4) must be a solution of (1.3). The aim of this paper is to give oscillation criteria for (1.1) and (1.2) which are similar to those given by Theorem A, but admit test functions which need not be a solution of (1.3) and the assumption $q(x) \leqslant 0$ is not still needed. Particularly, if $p(x) = x^\alpha$, $h(x) = x^{\sigma/2}$, our criteria comply with criteria given in Theorem B without the assumption $q(x) \leqslant 0$ (of course, similarly to some recent papers, e.g., [6], in this case lim inf in (1.6) must be replaced by lim sup).

The paper is organized as follows. In the next section we recall basic facts concerning solutions of (1.1) and corresponding linear Hamiltonian systems (LHS). Section 3 involves the main results of the paper—oscillation criteria for (1.2). In the last section we give remarks and comments concerning applications of oscillation criteria from Section 3 and their extension to equations of the form (1.1).

II. Auxiliary Results

Let y be a solution of (1.1). Set $u_i = y^{(i-1)}$, $i = 1, ..., n$, $v_n = p_n y^{(n)}$, $v_{n-i} = -v'_{n-i+1} + p_{n-i} y^{(n-i)}$, $i = 1, ..., n$, $u = (u_1, ..., u_n)$, $v = (v_1, ..., v_n)$. The n dimensional vectors u, v are solutions of the LHS

$$u' = Au + B(x)\, v, \qquad v' = C(x)\, u - A^T v, \tag{2.1}$$

where

$$\begin{aligned} B(x) &= \operatorname{diag}\{0, ..., 0, p_n^{-1}(x)\}, \\ C(x) &= \operatorname{diag}\{p_0(x), ..., p_{n-1}(x)\}, \\ A_{ij} &= \begin{cases} 1 & \text{for } j = i+1,\ i = 1, ..., n-1 \\ 0 & \text{elsewhere.} \end{cases} \end{aligned} \tag{2.2}$$

We say that the solution y of (1.1) *generates* the solution (u, v) of (1.4). Simultaneously with (1.4) consider the matrix system

$$U' = AU + B(x)\, V, \qquad V' = C(x)\, U - A^T V, \tag{2.3}$$

where U, V are $n \times n$ matrices. A *self-conjugate solution* of (2.3) (i.e., $U^T(x)\, V(x) \equiv V^T(x)\, U(x)$, alternative terminology is *self-conjoined* [12], *isotropic* [2], our terminology is due to [13]) is said to be *principal* (*nonprincipal*) at b if the matrix U is nonsingular near b and $\lim_{x \to b} (\int_d^x U^{-1}(s)\, B(s)\, U^{T-1}(s)\, ds)^{-1} = 0$ $(\lim_{x \to b} (\int_d^x U^{-1}(s)\, B(s) \times U^{T-1}(s)\, ds)^{-1} = M$, M being a nonsingular $n \times n$ matrix), for some $d \in I$, which is sufficiently close to b. A principal solution of (2.3) at b is determined uniquely up to a right multiply by a nonsingular $n \times n$ matrix. Let $y_1, ..., y_n$ be solutions of (1.1) and let $(u_1, v_1), ..., (u_n, v_n)$ be the solutions of (2.1) generated by $y_1, ..., y_n$. If the vectors $u_1, ..., u_n$, $v_1, ..., v_n$ form the columns of the solution (U, V) of (2.3) we say that this solution is generated by the solutions $y_1, ..., y_n$ of (1.1). A system of solutions $y_1, ..., y_n$ of (1.1) is said to be *principal* (*nonprincipal*) at b if the solution (U, V) of (2.3) generated by $y_1, ..., y_n$ is principal (nonprincipal) at b.

Two points $x_1, x_2 \in I$ are said to be *conjugate* relative to (1.1) if there exists a nontrivial solution y of (1.1) such that $y^{(i)}(x_1) = 0 = y^{(i)}(x_2)$, $i = 0, ..., n-1$. Equation (1.1) is said to be *conjugate* on an interval $I_0 \subseteq I$ if there exist $x_1, x_2 \in I_0$ which are conjugate relative to (1.1), in the opposite case (1.1) is said to *disconjugate*. Equation (1.1) is said to be *oscillatory* at b if for every $d \in I$ there exists a pair of distinct points $x_1, x_2 \in (d, b)$ which are conjugate, in the opposite case (1.1) is said to be *nonoscillatory* at b. Principal and nonprincipal system of solutions of (1.1) at b exist if this equation is nonoscillatory at b.

Let (U, V) be a self-conjugate solution of (2.3) such that U is nonsingular on some subinterval $I_0 \subseteq I$. Then

$$(U_1, V_1) = \left(U \int_d^x U^{-1} B U^{T-1}\, dt,\; V \int_d^x U^{-1} B U^{T-1}\, dt + U^{T-1} \right), \qquad d \in I$$

is also a self-conjugate solution of (2.3) and $W = VU^{-1}$ is a solution of the Riccati matrix differential equation

$$W' + A^T W + WA + WB(x)\, W - C(x) = 0. \tag{2.4}$$

If (U_b, V_b) is the principal solution of (2.3) at b, then the solution $W_b = V_b U_b^{-1}$ of (2.4) is said to be *distinguished* in b. Let $\tilde{W}$ be a solution of (2.4) which exists on an interval (d, b), $d \in I$, then $\tilde{W}(x) \geqslant W_b(x)$ on (d, b) (the inequality $\tilde{W} \geqslant W_b$ means that the matrix $\tilde{W} - W_b$ is nonnegative definite), see [2].

LEMMA 1 [9]. *Equation* (1.1) *is conjugate on* $I_0 = (c, d) \subseteq I$ *if and only if there exists a nontrivial function* $y \in \mathring{W}_n^2(I_0)$ ($\mathring{W}_n^2$ *is the Sobolev space of functions for which* $y, ..., y^{(n-1)}$ *are absolutely continuous in* I_0, $y_n \in \mathscr{L}^2(I_0)$ *and* supp $y \subset I_0$) *such that*

$$I(y; c, d) = \int_c^d \left[\sum_{k=0}^{n} p_k(x) (y^{(k)}(x))^2 \right] dx \leqslant 0.$$

LEMMA 2 [2]. *Let* (1.1) *be disconjugate on* $I_0 = (c, d) \subset I$ *and let* $x_1, x_2 \in I_0$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n) \in \mathbf{R}^n$ *be arbitrary. There exists a unique solution* y *of* (1.1) *such that*

$$y^{(i-1)}(x_1) = \alpha_i, \qquad y^{(i-1)}(x_2) = \beta_i, \qquad i = 1, ..., n.$$

LEMMA 3 [1]. *Let* $h \in C^{2n}(I)$ *be a positive real-valued function. The transformation* $y = hz$ *transforms Eq.* (1.1) *into the equation*

$$\sum_{k=0}^{n} (-1)^n (P_k(x)\, z^{(k)})^{(k)} = 0,$$

where $P_n = h^2 p_n$, $P_0 = \sum_{k=0}^{n} (-1)^k h (p_k h^{(k)})^{(k)}$.

LEMMA 4. *Let* (U_b, V_b) *be the principal solution at* b *of* (2.3) *with* $C \equiv 0$ (*this LHS corresponds to* (1.3)) *and*

$$(U_1, V_1) = \left(U_b \int_d^x U_b^{-1} B U_b^{T-1}\, dt,\; V_b \int_d^x U_b^{-1} B U_b^{T-1}\, dt + U_b^{T-1} \right), \qquad d \in I$$

$W_b = V_b U_b^{-1}$, $W_1 = V_1 U_1^{-1}$. *Then* $W_b \leqslant 0$ *and* $W_1 \geqslant 0$ *near* b.

Proof. Let D be the $n \times n$ matrix solution of the first order system $D' = AD$. Then $(D(x), 0)$, $(D(x) \int_d^x D^{-1} B D^{T-1}, D^{T-1}(x))$ form the base of the solution space of (2.3) (with $C \equiv 0$), hence $(U_1, V_1) = (DC_1 + D \int_d^x D^{-1} B D^{T-1} C_2\, dt, D^{T-1} C_2)$, where C_1, C_2 are constant $n \times n$ matrices. Substituting $x = d$ in the last equality we have $C_1 = 0$. Hence $V_1 U_1^{-1} = D^{T-1} (\int_d^x D^{-1} B D^{T-1}\, dt)^{-1}$ is nonnegative definite. Since $W \equiv 0$ is a soluton of (2.3) with $C \equiv 0$ which exists on the whole interval I, we have $W_b \leqslant 0$.

Besides the definition of oscillation and nonoscillation of (1.1) by means of the concept of conjugate points, we will use another definition of oscillation and nonoscillation of linear differential equations introduced by Nehari. A linear differential equation

$$y^{(n)} + q_{n-1}(x)\, y^{(n-1)} + \cdots + q_0(x)\, y = 0, \tag{2.5}$$

$q_k \in C^k(I)$, $k = 0, ..., n-1$, is said to be disconjugate on $I_0 \subseteq I$ in the sense of Nehari (shortly N-disconjugate) if any nontrivial solution of (2.5) has at most $n-1$ zeros on I_0, every zero point counted according to its multiplicity. Equation (2.5) is said to be eventually N-disconjugate at b if there exists $b_0 \in I$ such that (2.5) is N-disconjugate on (b_0, b).

A system of solutions $y_1, ..., y_n$ of (2.5) is said to form a Markov system of solutions on $I_0 \subseteq I$ if n Wronskians

$$W(y_1, ..., y_k) = \begin{vmatrix} y_1 & \cdots & y_k \\ \vdots & \vdots & \vdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix},$$

$k = 1, ..., n$, are positive throughout I_0. The system $y_1, ..., y_n$ is said to form a Descartes system of solutions on I_0 if all Wronskians $W(y_{i_1}, ..., y_{i_k})$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, $k = 1, ..., n$, are positive throughout I_0.

Recall briefly the relation between oscillation properties of (2.5) and the existence of a Markov or Descartes system of solutions of this equation (for the proofs of these statements see [2, Chap. III]).

LEMMA 5. *Equation* (2.5) *is N-disconjugate on $(b_0, b) \subseteq I$ if and only if there exists a Markov system of solutions of* (2.5) *on* (b_0, b). *This system can be found in such a way that it satisfies the additional conditions*

$$\begin{aligned} y_i &> 0 \qquad \text{on } (b_0, b),\ i = 1, ..., n, \\ y_{k+1} &= o(y_k) \qquad \text{for} \quad x \to b,\ k = 2, ..., n. \end{aligned} \tag{2.6}$$

Moreover, $y_1, ..., y_n$ form the Descartes system of solutions of (2.5) *near b.*

LEMMA 6. *Let* $y_1, \ldots, y_n$ *be a Descartes system of solutions of* (2.5) *satisfying* (2.6). *If* $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, $1 \leqslant j_1 < \cdots < j_k \leqslant n$, $1 \leqslant k \leqslant n$, *are distinct k-tuples such that* $i_l \leqslant j_l$, $l = 1, \ldots, k$, *then*

$$\lim_{x \to b} W(y_{i_1}, \ldots, y_{i_k})/W(y_{j_1}, \ldots, y_{j_k}) = 0.$$

LEMMA 7. *Let Eq.* (1.1) *be eventually N-disconjugate at b and let* $y_1, \ldots, y_{2n}$ *be a Descartes system of solutions of this equation satisfying* (2.6) (*with n replaced by* $2n$). *Then* $y_1, \ldots, y_n$ *form a principal system of solutions of* (1.1) *at b.*

LEMMA 8. *Let* (2.5) *be eventually N-disconjugate at b and let* $y_1, \ldots, y_n$ *be solutions of* (2.5) *such that* y_k *has zero multiplicity* $k-1$ *at* $c \in I$, $k = 1, \ldots, n$, *where c is sufficiently close to b. Then* $y_1, \ldots, y_n$ *is a Descartes system of functions on* (c, b) *satisfying* (2.6).

LEMMA 9. *Let* (2.5) *be eventually disconjugate at b,* $c \in I$ *is sufficiently close to b, and* $u_1, \ldots, u_n$ *are solutions of* (2.5) *satisfying conditions*

$$u_k^2(c) + (u_k'(c))^2 + \cdots + (u_k^{(n-1)}(c))^2 = 1,$$
$$u_k^{(j)}(t) = 0, \qquad j = 0, \ldots, n-k-1,\ (-1)^{n-k} u_k^{(n-k)}(t) > 0,$$
$$k = 1, \ldots, n.$$

Then the remaining initial conditions $u_k^{(j)}(t)$, $j = n-k+1, \ldots, n-1$ *can be chosen in such a way that the* $u_k^{(i)}$ *converge uniformly* (*as* $t \to b$) *on every compact subinterval of* $[c, b)$ *to functions* $y_k^{(i)}$, $i = 0, \ldots, n-1$, *where* $y_n, \ldots, y_1$ *is in the reversed order a Descartes system of solutions of* (2.5) *satisfying* (2.6).

III. OSCILLATION CRITERIA

Let $h \in C^{2n}(I)$ be a positive real-valued function and suppose that Eq. (1.1) is eventually N-disconjugate at b. The function h is said to be *compatible* with Eq. (1.1) if this function can be inserted into any Descartes system of solutions of (1.1) near b satisfying (2.6) (with $2n$ instead of n) in such a way that the new system of $2n+1$ functions is again a Descartes system of functions and satisfies (2.6) (with $2n+1$ instead of n). For example, the function $h(x) = x^\beta$, $\beta \notin \{0, \ldots, 2n-1\}$ is compatible with the equation $y^{(2n)} = 0$.

If h is compatible with (1.1), we say that this function grows near b *more rapidly* (*slowly*) *than principal* (*nonprincipal*) *solutions* of (1.1), if $y_k = o(h)$ as $x \to b$ ($h = o(\tilde{y}_k)$) for any solution y_k ($\tilde{y}_k$) of (1.1) which is involved in a principal (nonprincipal) system of solutions of (1.1) at b. In the previous example, if $\beta > n-1$ ($\beta < n$) then $h(x)$ grows near ∞ more rapidly (slowly) than principal (nonprincipal) solutions of $y^{(2n)}=0$.

THEOREM 1. *Let the following assumptions be satisfied.*

(i) *$y_1, ..., y_n$ is a principal system of solutions at b of (1.3), (U_b, V_b) is the principal solution at b of the associated matrix Hamiltonian system generated by $y_1, ..., y_n$, and $W_b = V_b U_b^{-1}$.*

(ii) *$h \in C^{2n}(I)$ is a positive real-valued function which is compatible with (1.3), grows more slowly at b than nonprincipal solutions of (1.3), $\int^b ph^{(n)2} < \infty$, and $\int^b qh^2 = \lim_{x \to b} \int^x qh^2$ exists and is finite.*

(iii) *The real-valued function $H(x) = \tilde{h}^T(x)\, U_b^{T-1}(x)(\int_d^x \tilde{B}(t)\, dt)^{-1} \times U_b^{-1}(x)\, \tilde{h}(x)$, where $\tilde{h}(x) = (h(x), h'(x), ..., h^{(n-1)}(x))^T$, $\tilde{B} = U_b^{-1} B U_b^{T-1}$, $B = \operatorname{diag}\{0, ..., 0, p^{-1}\}$, tends monotonically to 0 as $x \to b$.*

(iv) *It holds*

$$\limsup_{x \to b} \frac{\int_x^b ph^{(n)2} + \tilde{h}^T(x)\, W_b(x)\, \tilde{h}(x)}{H(x)} =: L > -1 \tag{3.1}$$

and

$$\limsup_{x \to b} \frac{\int_x^b qh^2}{H(x)} < -1 - L. \tag{3.2}$$

Then (1.2) is oscillatory at b.

Proof. According to Lemma 1, to prove the theorem it suffices to show that for any $c \in I$ there exists a nontrivial function $y \in \mathring{W}_n^2(c, b)$ such that

$$I(y; c, b) = \int_c^b (py^{(n)2} + qy^2)\, dx$$

is negative. Let $c < x_0 < x_1 < x_2 < x_3 < b$ and let f, g be solutions of (1.3) satisfying for $i = 0, ..., n-1$ the boundary conditions

$$f^{(i)}(x_0) = 0, \qquad f^{(i)}(x_1) = h^{(i)}(x_1), \qquad g(x_2) = h^{(i)}(x_2), \qquad g^{(i)}(x_3) = 0, \tag{3.3}$$

and define a test function y as

$$y = \begin{cases} 0, & x \in (c, x_0], \\ f(x), & x \in [x_0, x_1], \\ h(x), & x \in [x_1, x_2], \\ g(x), & x \in [x_2, x_3], \\ 0, & x \in [x_3, b). \end{cases}$$

In view of Lemma 2, the solutions f, g of (1.3) are by (3.3) determined uniquely and one can directly verify that f and g generate the solutions (u_1, v_1), (u_2, v_2) of the corresponding LHS which are of the form

$$u_1(x) = U_b(x)\left(\int_{x_0}^{x} \tilde{B}(s)\, ds\right)\left(\int_{x_0}^{x_1} \tilde{B}(s)\, ds\right)^{-1} U_b^{-1}(x_1)\, \tilde{h}(x_1),$$

$$v_1(x) = \left(V_b(x)\int_{x_0}^{x} \tilde{B}(s)\, ds + U_b^{T-1}(x)\right)$$

$$\times\left(\int_{x_0}^{x_1} \tilde{B}(s)\, ds\right)^{-1} U_b^{-1}(x_1)\, \tilde{h}(x_1),$$

$$u_2(x) = U_b(x)\left(\int_{x}^{x_3} \tilde{B}(s)\, ds\right)\left(\int_{x_2}^{x_3} \tilde{B}(s)\, ds\right)^{-1} U_b^{-1}(x_2)\, \tilde{h}(x_2),$$

$$v_2(x) = \left(V_b(x)\int_{x}^{x_3} \tilde{B}(s)\, ds - U_b^{T-1}(x)\right)$$

$$\times\left(\int_{x_2}^{x_3} \tilde{B}(s)\, ds\right)^{-1} U_b^{-1}(x_2)\, \tilde{h}(x_2),$$

where $\tilde{B} = U_b^{-1} B U_b^{T-1}$.

Computing the quadratic functional $I(y, c, d) = I(y; x_0, x_3)$, we have

$$\int_{x_0}^{x_1} pf^{(n)2} = \int_{x_0}^{x_1} v_1^T B v_1 = \int_{x_0}^{x_1} v_1^T (u_1' - Au_1)$$

$$= v_1^T u_1 \big|_{x_0}^{x_1} - \int_{x_1}^{x_1} (v_1'^T u_1 - v_1^T A u_1)$$

$$= v_1^T(x_1)\, u_1(x_1) - \int_{x_0}^{x_1} v_1^T (A - A)\, u_1$$

$$= \tilde{h}^T(x_1)\, U_b^{T-1}(x_1)\left(\int_{x_0}^{x_1} \tilde{B}\right)^{-1} U_b^{-1}(x_1)\, \tilde{h}(x_1)$$

$$+ \tilde{h}^T(x_1)\, W_b(x_1)\, \tilde{h}(x_1).$$

Similarly

$$\int_{x_2}^{x_3} pg^{(n)2} = \tilde{h}^T(x_2)\, U_b^{T-1}(x_2)\left(\int_{x_2}^{x_3} \tilde{B}\right)^{-1}$$

$$\times U_b^{-1}(x_2)\, \tilde{h}(x_2) - \tilde{h}(x_2)\, W_b(x_2)\, \tilde{h}(x_2).$$

Consequently

$$
\begin{aligned}
I(y; x_0, x_3) = {} & \tilde{h}^T(x_1)\, U_b^{T-1}(x_1) \left(\int_{x_0}^{x_1} \tilde{B} \right)^{-1} U_b^{-1}(x_1)\, \tilde{h}(x_1) \\
& + \tilde{h}^T(x_1)\, W_b(x_1)\, \tilde{h}(x_1) + \tilde{h}^T(x_2)\, U_b^{T-1}(x_2) \\
& \times \left(\int_{x_2}^{x_3} \tilde{B} \right)^{-1} U_b^{-1}(x_2)\, \tilde{h}(x_2) - \tilde{h}^T(x_2)\, W_b(x_2)\, \tilde{h}(x_2) \\
& + \int_{x_0}^{x_1} qf^2 + \int_{x_1}^{x_2} ph^{(n)2} + \int_{x_1}^{x_2} qh^2 + \int_{x_2}^{x_3} qg^2.
\end{aligned}
$$

In the next step we prove that the functions f/h, g/h are monotonic on (x_0, x_1) and (x_2, x_3), respectively. Having proved this monotonicity, using the second mean value of integral calculus we get $\int_{x_0}^{x_1} qf^2 = \int_{x_0}^{x_1} qh^2 (f/h)^2 = \int_{\xi_1}^{x_1} qh^2$, where $\xi_1 \in (x_0, x_1)$. Similarly $\int_{x_2}^{x_3} qg^2 = \int_{x_2}^{\xi_2} qh^2$, $\xi_2 \in (x_2, x_3)$. First consider the interval (x_0, x_1). Let $z_1, ..., z_n, z_{n+1}, ..., z_{2n}$ be a Descartes system of solutions of (1.3) (near b) which satisfies (2.6). The function h may be inserted into this system in such a way that the new system of $2n+1$ functions is again a Descartes system satisfying (2.6). Since h grows near b more slowly than nonprincipal solutions, by Lemma 7, $h/z_{n+i} \to 0$, $x \to b$, $i = 1, ..., n$. Suppose that h is inserted between z_{j-1} and z_j for some $j = 1, ..., n+1$. Then $w_i = -(z_i/h)'$, $i = 1, ..., j-1$, $w_i = (z_i/h)'$, $i = j, ..., 2n$ form a Markov system of functions near b. Indeed,

$$
\begin{aligned}
w_1 &= -(z_1/h)' = h^{-2} W(z_1, h) > 0, \\
W(w_1, w_2) &= W(-(z_1/h)', -(z_2, h)') = h^{-3} W(z_1, z_2, h) > 0
\end{aligned}
$$

and similarly $W(w_1, ..., w_k) > 0$, $k = 3, ..., 2n$. Consequently, $w_1, ..., w_{2n}$ is a fundamental system of solutions of a linear differential equation of the form (2.5), which is by Lemma 5 eventually N-disconjugate near b.

Now, to prove monotonicity of (f/h) on (x_0, x_1), it suffices to show that there exists $\delta > 0$ such that

$$
\operatorname{sgn}(f/h)'(x_0 + t) = \operatorname{sgn}(f/h)'(x_1 - t) \tag{3.4}
$$

for $t \in (0, \delta)$. Indeed, since the function $(f/h)'$ has zero points of multiplicity $n-1$ both in x_0 and x_1, and $w = (f/h)'$ is a solution of a $2n$ order equation which is N-disconjugate near b, (3.4) implies that $(f/h)'$ has an even number of zeros on (x_0, x_1). If $(f/h)'(t_0) = 0$ at some $t_0 \in (x_0, x_1)$ then $(f/h)'$ has at least $2n$ zeros on $[x_0, x_1]$ (counting multiplicity), which is a contradiction.

Using Taylor's expansion of $(f/h)'$ at x_0 and x_1 we have

$$(f/h)'(x)=\frac{(f/h)^{(n)}(x_0)}{(n-1)!}(x-x_0)^{n-1}+o((x-x_0)^{n-1}),$$

$$(f/h)'(x)=\frac{(f/h)^{(n)}(x_1)}{(n-1)!}(x-x_1)^{n-1}+o((x-x_1)^{n-1}),$$

hence (3.4) holds provided

$$\operatorname{sgn}(f/h)^{(n)}(x_0)=(-1)^{n-1}\operatorname{sgn}(f/h)^{(n)}(x_1).$$

Let $u_1, ..., u_n$ be solutions of (1.3) satisfying the conditions

$$u_i^{(j)}(x_0)=\begin{cases}0, & \text{for } j=0, ..., n-2+i,\\ 1, & \text{for } j=n-1+i,\end{cases}$$

$i=1, ..., n$. By Lemma 8 these solutions are contained in the subspace of the solution space of (1.3) spanned on nonprincipal solutions $z_{n+1}, ..., z_{2n}$ and $u_1, ..., u_n$ form a Descartes system near b. Without loss of generality one may suppose that $z_{n+1}, ..., z_{2n}$ are just equal to $u_1, ..., u_n$, i.e., $z_{n+i}=u_i$, $i=1, ..., n$. According to (3.3) the function f is of the form $f=c_1u_1+\cdots+c_nu_n$, where the constants $c_1, ..., c_n$ are solutions of a linear system

$$\sum_{i=1}^{n} c_i u_i^{(j)}(x_1)=h^{(j)}(x_1), \qquad j=0, ..., n-1,$$

hence by Cramer's rule

$$c_i=\frac{W(u_1, ..., u_{i-1}, h, u_{i+1}, ..., u_n)}{W(u_1, ..., u_n)}(x_1).$$

We have

$$\left(\frac{f}{h}\right)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}\left(\frac{1}{h}\right)^{(n-k)} \tag{3.5}$$

and this equality implies

$$\begin{aligned}(f/h)^{(n)}(x_0)&=h^{-1}(x_0)\,f^{(n)}(x_0)=h^{-1}(x_0)\,c_1\\ &=(h(x_0)\,W(u_1, ..., u_n)(x_1))^{-1}\cdot W(h, u_2, ..., u_n)(x_1)>0\end{aligned}$$

if x_1 is sufficiently close to b since $h, u_1, ..., u_n$ is a Descartes system near b. Similarly (3.5) gives

$$\begin{aligned}(f/h)^{(n)}(x_1) &= h^{-1}(x_1)(f^{(n)}(x_1) - h^{(n)}(x_1))\\ &= h^{-1}(x_1)\left(\sum_{i=1}^{u} c_i u_i^{(n)}(x_1) - h^{(n)}(x_1)\right)\\ &= (h(x_1)\, W(u_1, ..., u_n)(x_1))^{-1}\\ &\quad\times\left[\sum_{i=1}^{n} u_i^{(n)}(x_1)\, W(u_1, ..., u_{i-1}, h, u_{i+1}, ..., u_n)(x_1)\right.\\ &\qquad \left. - h^{(n)}(x_1)\, W(u_1, ..., u_n)(x_1)\right]\\ &= -(h(x_1)\, W(u_1, ..., u_n)(x_1))^{-1}\, W(u_1, ..., u_n, h)\\ &= (-1)^{n-1}\,(h(x_1)\, W(u_1, ..., u_n)(x_1))^{-1}\cdot W(h, u_1, ..., u_n)(x_1).\end{aligned}$$

Hence $\operatorname{sgn}(f/h)^{(n)}(x_1) = (-1)^{n-1}$ which we needed to prove.

Concerning the monotonicity of (g/h) on (x_2, x_3), we proceed as follows. Let $\tilde{u}_1, ..., \tilde{u}_n$ be solutions of (1.3) satisfying the conditions

$$\tilde{u}_{n+1-i}^{(j)}(x_3) = \begin{cases} 0, & j = 0, ..., n-2+i \\ (-1)^{n-1+i}, & j = n-1+i, \end{cases}$$

$i = 1, ..., n$. By Lemma 9, the remaining initial conditions at $x = x_3$ can be chosen in such a way that $\tilde{u}_i$ tend locally uniformly on (x_2, b) (as $x_3 \to b$) to functions which form a principal system and a Descartes system at b, hence, without loss of generality one may suppose that $\tilde{u}_i^{(j)} \rightrightarrows z_i^{(j)}$, $j = 0, ..., n-1$. Similarly as for $x \in (x_0, x_1)$ we have $g = \sum_{i=1}^{n} \tilde{c}_i \tilde{u}_i$, where

$$\tilde{c}_i = W(\tilde{u}_1, ..., \tilde{u}_{i-1}, h, \tilde{u}_{i+1}, ..., \tilde{u}_n)(x_2)/W(\tilde{u}_1, ..., \tilde{u}_n)(x_2)$$

and hence

$$\begin{aligned}(g/h)^{(n)}(x_3) &= h^{-1}(x_3)\, g^{(n)}(x_3) = h^{-1}(x_3)\, \tilde{c}_n\\ &= (h(x_3)\, W(\tilde{u}_1, ..., \tilde{u}_n)(x_2))^{-1}\, W(\tilde{u}_1, ..., \tilde{u}_{n-1}, h)(x_2),\\ (g/h)^{(n)}(x_2) &= h^{-1}(x_2)\,(g^{(n)}(x_2) - h^{(n)}(x_2))\\ &= -(h(x_3)\, W(\tilde{u}_1, ..., \tilde{u}_n)(x_2))^{-1}\, W(\tilde{u}_1, ..., \tilde{u}_n, h)(x_2).\end{aligned}$$

Denote

$$\mathbb{K} = \{i \in \{1, ..., n\}, \lim_{x \to b} (z_i/h) = 0\}$$

and $k = \max \mathbb{K}$ if $\mathbb{K} \neq \emptyset$ and $k = 0$ if $\mathbb{K} = \emptyset$. If $k = n$, we have $\operatorname{sgn}(g/h)^{(n)}(x_2) = (-1)^{n-1} \operatorname{sgn}(g/h)^{(n)}(x_2)$ and by the same argument as above g/h is monotonic on (x_2, x_3). If $k < n$, it is not difficult to verify that $\operatorname{sgn}(g/h)^{(n)}(x_2) = (-1)^n \operatorname{sgn}(g/h)^{(n)}(x_2)$, i.e., the monotonicity of (g/h) cannot be proved by the above used method. To overcome this difficulty, we proceed in the following way. We replace the function g by a function $\tilde{g}$ which is a solution of the equation

$$(\tilde{p}(x)\, y^{(n)})^{(n)} = 0 \tag{3.6}$$

satisfying the same boundary conditions as g and $\tilde{p} \in C^n$ is a function for which $\tilde{p}(x) \geqslant p(x)$ near b, such that h is contained in a principal system of solutions of (3.6).

The function $\tilde{p}$ we construct as follows. Let M be the subspace of the solution space of (1.3) consisting of solutions y for which $y^{(n)} \not\equiv 0$ (such solutions are sometimes called *nonpolynomial*) and let $\tilde{y}_1, ..., \tilde{y}_n$ be the base of M such that $\tilde{y}_1, ..., \tilde{y}_n$ is a Descartes system of functions for which (2.6) holds. Denote $m = \min\{i \in \{1, ..., n\}, \lim \tilde{y}_i/h = \infty\}$ (the last set is nonempty, otherwise h would grow more fastly than n solutions of (1.3) which contradicts the fact that $k < n$). By Lemma 6 we have

$$\lim_{x \to b} \frac{W(1, x, ..., x^{n-1}, \tilde{y}_j)}{W(1, x, ..., x^{n-1}, h)} = \lim_{x \to b} \frac{\tilde{y}_j^{(n)}}{h^{(n)}} = \begin{cases} 0 & \text{for } j < m \\ \infty & \text{for } j \geqslant m. \end{cases}$$

Set

$$\tilde{p} = \frac{p\tilde{y}_m^{(n)}}{h^{(n)}}.$$

Then $\lim_{x \to b} \tilde{p}/p = \infty$, i.e., $\tilde{p} > p$ near b and since $py_m^{(n)}$ is a polynomial, h is a solution of (3.6) which, by definition of m, belongs to the principal system of solutions of (3.6) at b.

The transformation $y = hz$ transforms (3.6) into the equation

$$\sum_{k=0}^{n} (-1)^k (P_k z^{(k)})^{(k)} = 0,$$

where $P_0 = (-1)^n h(\tilde{p}h^{(n)})^{(n)} = 0$ (Lemma 3), i.e., $w = (\tilde{g}/h)'$ is a solution of a $2n-1$ order linear equation which is by the same argument as above eventually N-disconjugate at b. Since w has zeros of multiplicity $n-1$ at $x = x_2$ and $x = x_3$, w does not vanish on (x_2, x_3), i.e., $\tilde{g}/h$ is monotonic on this interval. We have

$$\int_{x_2}^{x_3} p\tilde{g}^{(n)2} \leqslant \int_{x_2}^{x_3} \tilde{p}\tilde{g}^{(n)2}$$
$$= \tilde{h}^T(x_2)\, \bar{W}_b(x_2)\, \tilde{h}(x_2) + \tilde{h}^T(x_2)\, U_b^{T-1}(x_2)$$
$$\times \left(\int_{x_2}^{x_3} \bar{U}_b^{T-1} \bar{B} \bar{U}_b^{-1}\right) \bar{U}_b^{-1}(x_2)\, \tilde{h}(x_2),$$

where $(\bar{U}_b, \bar{V}_b)$ is the principal solution of the LHS corresponding to (3.6), $\bar{W}_b = \bar{V}_b \bar{U}_b^{-1}$ and $\bar{B} = \operatorname{diag}\{0, ..., 0, \tilde{p}^{-1}\}$.

Now return to the computation of $I(y; x_0, x_3)$. First consider the case when we do not need to replace g by $\tilde{g}$. We have

$$I(y; x_0, x_3) = \tilde{h}^T(x_1)\, U_b^{T-1}(x_1) \left(\int_{x_0}^{x_1} \tilde{B}\right)^{-1} U_b^{-1}(x_1)\, \tilde{h}(x_1)$$
$$+ \tilde{h}^T(x_1)\, W_b(x_1)\, \tilde{h}(x_1) + \int_{x_1}^{x_2} ph^{(n)2}$$
$$+ \tilde{h}^T(x_2)\, U_b^{T-1}(x_2) \left(\int_{x_2}^{x_3} \tilde{B}\right)^{-1} U_b^{-1}(x_2)\, \tilde{h}(x_2)$$
$$- \tilde{h}^T(x_2)\, W_b(x_2)\, \tilde{h}(x_2) + \int_{\xi_1}^{\xi_2} qh^2$$
$$\leqslant H(x_1) \Bigg[\tilde{h}^T(x_1)\, U_b^{T-1}(x_1) \left(\int_{x_0}^{x_1} \tilde{B}\right)^{-1} U_b^{-1}(x_1)\, \tilde{h}(x_1)/H(x_1)$$
$$+ \left(\int_{x_1}^{x_2} ph^{(n)2} + \tilde{h}^T(x_1)\, W_b(x_1)\, \tilde{h}(x_1)\right) \Big/ H(x_1)$$
$$+ \int_{\xi_1}^{\xi_2} qh^2/H(\xi_1) + \tilde{h}^T(x_2)\, U_b^{T-1}(x_2) \left(\int_{x_2}^{x_3} \tilde{B}\right)^{-1} U_b^{-1}(x_2)$$
$$\times \tilde{h}(x_2)/H(x_1) - \tilde{h}^T(x_2)\, W_b(x_2)\, \tilde{h}(x_2)/H(x_1) \Bigg].$$

Denote

$$(U_1, V_1) = \left(U_b \int_d^x \tilde{B},\ V_b \int_d^x \tilde{B} + U_b^{T-1}\right),$$
$$W_1 = V_1 U_1^{-1} = W_b + U_b^{T-1} \left(\int_d^x \tilde{B}\right)^{-1} U_b^{-1}.$$

By Lemma 4, $W_1 \geqslant 0$, $W_b \leqslant 0$ near b, hence $0 \leqslant -\tilde{h}^T W_b \tilde{h} = -h^T W_1 \tilde{h} + H(x) \leqslant H(x)$, consequently

$$\lim_{x \to b} \tilde{h}^T(x)\, W_b(x)\, \tilde{h}(x) = 0. \tag{3.7}$$

Let $\varepsilon > 0$ be such that lim sup in (3.2) is less than $-1 - L - 6\varepsilon$. Since (U_b, V_b) is the principal solution at b, all eigenvalues of the matrix $\int_{x_0}^{x} \tilde{B}$ tend to ∞ as $x \to b$, hence $\varepsilon \int_{x_0}^{x} \tilde{B} > \int_{d}^{x_0} \tilde{B}$ if x is sufficiently close to b. It follows $(\int_{x_0}^{x} \tilde{B})^{-1} < (1+\varepsilon)(\int_{d}^{x} \tilde{B})^{-1}$ and hence

$$\frac{\tilde{h}^T(x)\, U_b^{T-1}(x)\, (\int_{x_0}^{x} \tilde{B})^{-1}\, U_b^{-1}(x)\, \tilde{h}(x)}{H(x)} < 1 + \varepsilon.$$

Choose x_0 such that all statements in the previous parts of the proof required or claimed to hold near b hold on (x_0, b) and $\int_x^b qh^2/H(x) < -1 - L - 5\varepsilon$ whenever $x > x_0$. Choose the number $x_1 > x_0$ such that

$$\frac{\tilde{h}^T(x_1)\, W_b(x_1)\, \tilde{h}(x_1) + \int_{x_1}^{b} ph^{(n)2}}{H(x_1)} < L + 2\varepsilon$$

and $x_2 > x_1$ can be chosen such that $[\tilde{h}^T(x_1)\, W_b(x_1)\, \tilde{h}(x_1) + \int_{x_1}^{x_2} ph^{(n)}]/H(x_1) < L + \varepsilon$, $-\tilde{h}^T(x_2)\, W_b(x_2)\, \tilde{h}(x_2)/H(x_1) < \varepsilon$ (see (3.7)) and $\int_{x_1}^{t} qh^2/H(x_1) < -1 - L - 4\varepsilon$ whenever $t > x_2$. Finally, since (U_b, V_b) is a principal solution, $x_3 > x_2$ can be chosen such that $\tilde{h}^T(x_2)\, U_b^{T-1}(x_2)\, (\int_{x_2}^{x_3} \tilde{B})^{-1}\, U_b^{-1}(x_2)\, h(x_2)/H(x_1) < \varepsilon$. Summarizing all estimates, we have

$$\begin{aligned} I(y; x_0, x_3) \leqslant H(x) \Bigg[& \tilde{h}^T(x_1)\, U_b^{T-1}(x_1) \left(\int_{x_0}^{x_1} \tilde{B} \right)^{-1} U_b^{-1}(x_1)\, \tilde{h}(x_1)/H(x_1) \\ & + \left(\int_{x_1}^{x_2} ph^{(n)2} + \tilde{h}^T(x_1)\, W_b(x_1)\, h(x_1) \right) \Big/ H(x_1) \\ & + \int_{\xi_1}^{\xi_2} qh^2/H(\xi_1) + \tilde{h}_2^T(x_2)\, U_b^{T-1}(x_2) \left(\int_{x_2}^{x_3} \tilde{B} \right)^{-1} \\ & \times U_b^{-1}(x_2)\, \tilde{h}(x_2)/H(x_1) - \tilde{h}^T(x_2)\, W_b(x_2)\, \tilde{h}(x_2)/H(x_1) \Bigg] \\ \leqslant H(x_1)\, & [(1+\varepsilon) + (L+\varepsilon) + (-1 - L - 4\varepsilon) + \varepsilon + \varepsilon] \leqslant 0. \end{aligned}$$

Let us finish the proof with the case when $y = \tilde{g}$ for $x \in (x_2, x_3)$, $\tilde{g}$ being the solution of (3.6) satisfying (3.3). We have

$$\begin{aligned} \int_{x_2}^{x_3} \tilde{p}\tilde{g}^{(n)2} = {} & \tilde{g}_1^T(x)\, \tilde{W}_b(x)\, \tilde{g}_1(x) \big|_{x_2}^{x_3} + \tilde{g}_1^T(x_2)\, \tilde{U}_b^{T-1}(x_2) \\ & \times \left(\int_{x_2}^{x_3} \tilde{U}_b^{-1} \tilde{B} \tilde{U}_b^{T-1} \right)^{-1} U_b^{-1}(x_2)\, \tilde{g}_1(x_2), \end{aligned}$$

where $(\tilde{g}_1, \tilde{g}_2)$ is the solution of the LHS corresponding to (3.6) generated by $\tilde{g}$, $\tilde{W}_b$ is the distinguished solution of the Riccati matrix equation associated with this LHS, i.e., $\tilde{W}_b = \tilde{V}_b \tilde{U}_b^{-1}$, where $(\tilde{U}_b, \tilde{V}_b)$ is the principal solution at b of the matrix LHS corresponding to (3.6). Since h is contained in the principal system of solutions of (3.6), there exists $c \in \mathbf{R}^n$ such that $\tilde{h}(x) = \tilde{U}_b(x)\, c$, hence

$$\begin{aligned} \tilde{g}_1^T(x_1)\, \tilde{W}_b(x)\, \tilde{g}_1(x)\big|_{x_2}^{x_3} &= -\tilde{h}^T(x_2)\, \tilde{W}_b(x_2)\, \tilde{h}(x_2) \\ &= -c^T \tilde{U}_b^T(x_2)\, \tilde{V}_b(x_2)\, c. \end{aligned}$$

By a direct computation one may verify that $\lim_{x \to b} c^T \tilde{U}_b^T(x)\, \tilde{V}_b(x)\, c = 0$ (cf. [3]) and since $(\tilde{U}_b, \tilde{V}_b)$ is the principal solution, $\lim_{x \to b} (\int_{x_2}^x \tilde{U}_b^{-1} \tilde{B} \tilde{U}_b^{T-1}) = 0$. Consequently, if x_2, x_3 are sufficiently close to b, we have

$$\int_{x_2}^{x_3} p\tilde{g}^{(n)2}/H(x_1) \leqslant \int_{x_2}^{x_3} \tilde{p}\tilde{g}^{(n)2}/H(x_1) < \varepsilon.$$

Since all other estimates are the same as in the previous case, the proof is now complete.

Similar to the second part of Theorem A, if h grows near b more rapidly than principal solutions of (1.3), we have the following statement.

THEOREM 2. *Let the following assumptions be satisfied.*

(i) $y_1, \ldots, y_n$ *is a nonprincipal system of solutions of* (1.3) *at* b, (U, V) *is the solution of the corresponding matrix* LHS *generated by* $y_1, \ldots, y_n$ *and* $W = VU^{-1}$.

(ii) $h \in C^{2n}(c, b)$, $c \in I$, *is a positive real-valued function which is compatible with* (1.3) *and grows near* b *more rapidly than principal solutions of* (1.3).

(iii) *The real-valued function* $H(x) = \tilde{h}(x)\, U^{T-1}(x)(\int_x^b \tilde{B})^{-1}\, U^{-1}(x) \times \tilde{h}(x)$ *tends monotonically to* ∞ *as* $x \to b$, *where* $\tilde{h} = (h, h', \ldots, h^{(n-1)})^T$, $\tilde{B} = U^{-1} B U^{T-1}$, $B = \operatorname{diag}\{0, \ldots, 0, p^{-1}\}$.

(iv) *It holds*

$$\limsup_{x \to b} \frac{\int_d^x p h^{(n)2} - \tilde{h}^T(x)\, W(x)\, \tilde{h}(x)}{H(x)} =: L > -1, \qquad d \in I \tag{3.8}$$

and

$$\limsup_{x \to b} \frac{\int_d^x q h^2}{H(x)} < -1 - L. \tag{3.9}$$

Then Eq. (1.2) *is oscillatory at* b.

Proof. The proof is similar to that of Theorem 1. Let the test function be the same as in this theorem. The solutions (u_1, v_1), (u_2, v_2) of the LHS corresponding to (1.3) generated by f and g can also be expressed in the form

$$u_1 = U\left(\int_{x_0}^{x} \tilde{B}\right)\left(\int_{x_0}^{x_1} \tilde{B}\right)^{-1} U^{-1}(x_1)\,\tilde{h}(x_1),$$

$$v_1 = \left(V\left(\int_{x_0}^{x} \tilde{B}\right) + U^{T-1}\right)\left(\int_{x_0}^{x_1} \tilde{B}\right)^{-1} \tilde{h}(x_1),$$

$$u_2 = U\left(\int_{x}^{x_3} \tilde{B}\right)\left(\int_{x_2}^{x_3} \tilde{B}\right)^{-1} U^{-1}(x_2)\,\tilde{h}(x_2),$$

$$v_2 = \left(V\left(\int_{x}^{x_3} \tilde{B}\right) - U^{T-1}\right)\left(\int_{x_2}^{x_3} \tilde{B}\right)^{-1} \tilde{h}(x_2)$$

and then

$$\begin{aligned} I(y; x_0, x_3) &= \int_{x_0}^{x_1} pf^{(n)2} + \int_{x_1}^{x_2} ph^{(n)2} + \int_{x_2}^{x_3} pg^{(n)2} \\ &\quad + \int_{x_0}^{x_1} qf^2 + \int_{x_1}^{x_2} qh^2 + \int_{x_2}^{x_3} qg^2 \\ &= H(x_2)\left[\left(\int_{x_0}^{x_1} pf^{(n)2} + \int_{x_0}^{x_1} qf^2\right)\Big/ H(x_2)\right. \\ &\quad + \left(\int_{x_1}^{x_2} ph^{(n)2} - \tilde{h}^T(x_2)\, W(x_2)\, h(x_2)\right)/H(x_2) \\ &\quad + \tilde{h}^T(x_2)\, U^{T-1}(x_2)\left(\int_{x_2}^{x_3} \tilde{B}\right)^{-1} U^{-1}(x_2)\,\tilde{h}(x_2)/H(x_2) \\ &\quad \left. + \left(\int_{x_1}^{x_2} qh^2 + \int_{x_2}^{x_3} qg^2\right)\Big/ H(x_2)\right]. \end{aligned}$$

Here we do not need the monotonicity of f/h on (x_0, x_1) and the monotonicity of g/h on (x_2, x_3) may be proved by the same method as monotonicity of f/h on (x_0, x_1) in the proof of Theorem 1. Consequently, using the second mean value theorem of integral calculus we have

$$
\begin{aligned}
I(y; x_0, x_3) \leqslant H(x_2) \Bigg[& \left(\int_{x_0}^{x_1} p f^{(n)2} + \int_{x_0}^{x_1} q f^2 \right) \Big/ H(x_2) \\
& + \left(\int_d^{x_2} p h^{(n)2} - \tilde{h}^T(x_2)\, W(x_2)\, \tilde{h}(x_2) \right) \Big/ H(x_2) \\
& - \int_d^{x_1} p h^{(n)2} / H(x_2) + \tilde{h}^T(x_2)\, U^{T-1}(x_2) \\
& \times \left(\int_{x_2}^{x_3} \tilde{B} \right)^{-1} U^{-1} \tilde{h}(x_2)/H(x_2) + \int_{x_1}^{\xi} q h^2 / H(\xi) \Bigg],
\end{aligned}
$$

$\xi \in (x_2, x_3)$, where the nonpositivity of $\int^x qh^2$ and monotonicity of $H(x)$ have been used.

Now, let $x_0, x_1 \in I$, $x_0 < x_1$ be arbitrary and let $\varepsilon > 0$ be such that the lim sup in (3.9) is less than $-1 - L - 5\varepsilon$. If $x_2 < x_3$ are sufficiently close to b, we have

$$
\left(\int_{x_0}^{x_1} p f^{(n)2} + \int_{x_0}^{x_1} q f^2 \right) \Big/ H(x_2) < \varepsilon,
$$

$$
\left(\int_{x_1}^{x_2} p h^{(n)2} - \tilde{h}^T(x_2)\, W(x_2)\, \tilde{h}(x_2) \right) \Big/ H(x_2) < L + \varepsilon,
$$

$$
\tilde{h}^T(x_2)\, U^{T-1}(x_2) \left(\int_{x_2}^{x_3} \tilde{B} \right) U^{-1}(x_2)\, \tilde{h}(x_2)/H(x_2) < 1 + \varepsilon,
$$

$$
- \int_d^{x_1} p h^{(n)2} / H(x_2) < \varepsilon
$$

and

$$
\int_d^{\xi} q h^2 / H(\xi) < -1 - L - 4\varepsilon,
$$

whenever $\xi \leqslant x_2$. Hence

$$
I(y; x_0, x_3) \leqslant H(x_2)[\varepsilon + (L + \varepsilon) + \varepsilon + (1 + \varepsilon) + (-1 - L - 4\varepsilon)] \leqslant 0.
$$

IV. Remarks

(i) If $q(x) \leqslant 0$, Theorem 1 may be reformulated as follows. Similarly one may reformulate Theorem 2.

THEOREM 3. *Suppose that Eq.* (1.1) *is nonoscillatory at b, $y_1, ..., y_n$ is a principal system of solutions of* (1.1) *at b, (U, V) is the solution of the matrix LHS corresponding to* (1.1) *generated by $y_1, ..., y_n$, and $W = VU^{-1}$. If $h \in C^{2n}$ is a positive real-valued such that*

$$\int^b \left(\sum_{k=0}^{n} p_k h^{(k)2} + qh^2 \right) < \infty, \tag{4.1}$$

$$\lim_{x \to b} \tilde{h}^T(x)\, W(x)\, \tilde{h}(x) = 0, \tag{4.2}$$

$$\liminf_{x \to b} \frac{\int_x^b (\sum_{k=0}^n p_k h^{(k)2} + gh^2) + \tilde{h}^T(x)\, W(x)\, h(x)}{H(x)} < -1, \tag{4.3}$$

where $\tilde{h}$, H are the same as in Theorem 1, *then the equation*

$$\sum_{k=0}^{n} (-1)^k\, (p_k\, y^{(k)})^{(k)} + qy = 0 \tag{4.4}$$

is oscillatory at b.

Proof. Let the test function y be the same as in Theorem 1 but f, g are solutions of (1.1) (instead of (1.3)) satisfying (3.3). Then

$$\begin{aligned}\int_{x_0}^{x_3} \left(\sum_{k=0}^{n} p_k\, y^{(k)2} + qy^2 \right) \leqslant{} & \tilde{h}^T(x_1)\, U^{T-1}(x_1) \left(\int_{x_0}^{x_1} \tilde{B} \right)^{-1} U^{-1}(x_1)\, \tilde{h}(x_1) \\ & + \tilde{h}^T(x_1)\, W(x_1)\, \tilde{h}(x_1) + \int_{x_1}^{x_2} \sum_{k=0}^{n} p_k h^{(k)2} + \int_{x_1}^{x_2} qh^2 \\ & + \tilde{h}^T(x_2)\, U^{T-1}(x_2) \left(\int_{x_2}^{x_3} \tilde{B} \right)^{-1} \\ & \times U^{-1}(x_2)\, \tilde{h}(x_2) - \tilde{h}^T(x_2)\, W(x_2)\, \tilde{h}(x_2).\end{aligned}$$

Now, if (4.1) holds, using (4.2) and (4.3) one may prove similarly as in Theorem 1 that $\int_{x_0}^{x_3} (\sum_{k=0}^n p_k\, y^{(k)2} + qy^2) \leqslant 0$ if $x_0 < x_1 < x_2 < x_3 < b$ are sufficiently close to b.

(ii) Observe the following fact. In Theorem 1 we have needed the monotonicity of the function (y/h) both on intervals (x_0, x_1), (x_1, x_3) (in order to remove the assumption $q(x) \leqslant 0$ near b), in contrast to Theorem 2, where only the monotonicity of (y/h) on (x_0, x_1) has been used. This asymmetry is unnatural and Theorem 1 may be probably proved without monotonicity of (y/h) on (x_2, x_3). A closer examination of the proof of Theorem 2 also reveals the fact that all arguments can be used in order to prove the oscillation of the more general Eq. (4.4) if Eq. (1.1) is eventually N-disconjugate at b. Indeed, all we need to prove monotonicity of the function y/h on (x_2, x_3) is the existence of a Descartes system of solutions into which the function h may be "compatibly" inserted and the existence of this system of solutions is just guaranteed by the eventual N-disconjugacy of (1.1). On the other hand, the proof of Theorem 1 presented here relies on the fact that we consider only the equation of the form (1.2) (Lemma 4 and the construction of y on (x_2, x_3)).

(iii) If $p(x) = x^\alpha$, $\alpha \notin \{1, ..., 2n-1\}$, $h(x) = x^{\sigma/2}$, $\sigma \in \mathbf{R}$, the test function y from the proofs of Theorems 1, 2 is the same as the test function used in Theorem B, so this theorem remains valid without the assumption $q(x) \leqslant 0$ for large x if lim sup in $(1.6)_{1,2}$ is replaced by lim inf. Moreover, the test functions like $h(x) = x^{\sigma/2}(\ln_k x)^\beta$, where $\beta \in \mathbf{R}$ and $\ln_k$ is the k-times iterated logarithm, are also compatible with the equation $(x^\alpha y^{(n)})^{(n)} = 0$ and can be used as test functions in oscillation criteria which do not require non-positivity of $q(x)$.

(iv) The oscillation criteria given in Theorem A may be, roughly speaking, formulated as follows. Let h be a solution of (1.3) which is involved in a principal system of solutions and $\int_x^b qh^2$ does not tend to zero as $x \to b$ too rapidly (i.e., $(1.4)_1$ holds) then (1.2) is oscillatory at b. If h is a solution of (1.3) involved in a nonprincipal system and $\int^x qh^2$ tends to $-\infty$ (as $x \to b$) sufficiently rapidly (i.e., $(1.4)_2$ holds) then (1.2) is also oscillatory at b. The case when $\int^b qh^2 = -\infty$ and h is involved in a principal system of solutions covers the following criterion introduced in [5] (the oscillation criteria of this kind are sometimes called "Leighton–Wintner type" criteria).

THEOREM C. *Let h be a solution of* (1.3) *which is involved in a principal system of solutions at b. If*

$$\int^b qh^2 = \lim_{x \to b} \int^x qh^2 = -\infty \tag{4.5}$$

then (1.2) *is oscillatory at b.*

It is natural to try to replace the assumption "h is involved in a principal system of solutions at b" by the assumption "h is compatible with (1.3) and grows near b more slowly than nonprincipal solutions." As the result we have the following statement (which may be proved similarly as Theorem 1).

THEOREM 4. *Let $h \in C^{2n}$ be a positive real-valued function which is compatible with* (1.3) *and grows near b more slowly than nonprincipal solutions of* (1.3). *If*

$$\lim_{x \to b} \sup_{t \geq x} [\bar{h}^T(t)\, \tilde{h}(t) - \tilde{h}^T(x)\, W(x)\, \tilde{h}(x)] < \infty, \tag{4.6}$$

where $\tilde{h} = (h, h', ..., h^{(n-1)})^T$, $\bar{h} = ((-1)^{n-1}(ph^{(n)})^{(n-1)}, (-1)^{n-2}(ph^{(n)})^{(n-2)}, ..., ph^{(n)})^T$, *$W$ is the distinguished solution at b of the Riccati matrix equation associated with* (1.3), *and*

$$\int^b h[(-1)^n (ph^{(n)})^{(n)} + qh] = -\infty,$$

then (1.2) *is oscillatory at b.*

Note that in the setting of Theorem C the assumption (4.6) is automatically satisfied. It would be interesting to compare Theorem 4 (especially condition (4.6)) with the following second order oscillation criterion [2, Chap. I].

THEOREM D. *Let $h \in C^2$ be a positive real-valued function for which*

$$\int^b p^{-1}h^{-2} = \infty, \qquad \int^b h[-(ph')' + qh] = -\infty.$$

Then the second order equation $-(py')' + qy = 0$ is oscillatory at b.

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